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Generic convergence of infinite products of nonexpansive mappings with unbounded domains

*Simeon Reich and Alexander J. Zaslavski **

Department of Mathematics, Technion – Israel Institute of Technology, Haifa, Israel

We study the generic convergence of infinite products of nonexpansive mappings with unbounded domains in hyperbolic metric spaces.

Keywords: fixed point, generic property, hyperbolic metric space, infinite product, nonexpansive mapping

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****Correspondence:***
Alexander J. Zaslavski,
Department of Mathematics, Technion – Israel Institute of Technology, Amado Mathematics Building, Haifa 32000, Israel
ajzasl@techunix.technion.ac.il

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1. Introduction and the Main Result

Let (X, ρ) be a metric space and let R^1 denote the real line. We say that a mapping $c : R^1 \to X$ is a *metric embedding* of R^1 into X if $\rho(c(s), c(t)) = |s - t|$ for all real s and t. The image of R^1 under a metric embedding will be called a *metric line*. The image of a real interval $[a, b] = \{t \in R^1 : a \le t \le b\}$ under such a mapping will be called a *metric segment*.

Assume that (X, ρ) contains a family M of metric lines such that for each pair of distinct points x and y in X, there is a unique metric line in M which passes through x and y. This metric line determines a unique metric segment joining x and y. We denote this segment by $[x, y]$. For each $0 \le t \le 1$, there is a unique point z in $[x, y]$ such that

$$\rho(x, z) = t\rho(x, y) \text{ and } \rho(z, y) = (1 - t)\rho(x, y).$$

This point is denoted by $(1 - t)x \oplus ty$. We say that X, or more precisely, (X, ρ, M), is a *hyperbolic metric space* if

$$\rho(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}x \oplus \frac{1}{2}z) \le \frac{1}{2}\rho(y, z)$$

for all x, y, and z in X. An equivalent requirement is that

$$\rho(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}w \oplus \frac{1}{2}z) \le \frac{1}{2}(\rho(x, w) + \rho(y, z))$$

for all x, y, z, and w in X. A set $K \subset X$ is called *ρ-convex* if $[x, y] \subset K$ for all x and y in K.

It is clear that all normed linear spaces are hyperbolic in this sense. A discussion of more examples of hyperbolic spaces and, in particular, of the Hilbert ball can be found, for example, in Goebel and Reich [1] and Reich and Shafrir [2].

Let (X, ρ, M) be a complete hyperbolic metric space, and let $K \subset X$ be a nonempty, closed and ρ-convex subset of (X, ρ). For each $C : K \to K$, set $C^0(x) = x$ for all $x \in K$. Denote by $\mathcal{M}$ the set of all sequences $\{A_t\}_{t=1}^{\infty}$ of mappings $A_t : K \to K$, $t = 1, 2, \ldots$, such that for all integers $t \ge 1$,

$$\rho(A_t(x), A_t(y)) \le \rho(x, y) \text{ for all } x, y \in K. \qquad (1.1)$$

For each $x \in X$ and each $r > 0$, set

$$B(x, r) = \{y \in X : \ \rho(x, y) \le r\} \ \text{and} \ B_K(x, r) = B(x, r) \cap K.$$

Fix $\theta \in K$. For each $M, \epsilon > 0$, set

$$\begin{aligned}\mathcal{U}(M, \epsilon) = \{(\{A_t\}_{t=1}^{\infty}, \{B_t\}_{t=1}^{\infty}) \in \mathcal{M} \times \mathcal{M}:\\ \rho(A_t(x), B_t(x)) \le \epsilon \text{ for all } x \in B_K(\theta, M) \text{ and all integers}\\ t \ge 1\}.\end{aligned} \tag{1.2}$$

We equip the set $\mathcal{M}$ with the uniformity which has the base

$$\{\mathcal{U}(M, \epsilon) : M, \epsilon > 0\}.$$

It is not difficult to see that the uniform space $\mathcal{M}$ is metrizable (by a metric d) and complete.

Denote by $\mathcal{M}_*$ the set of all $\{A_t\}_{t=1}^{\infty} \in \mathcal{M}$ for which there exists a point $\tilde{x} \in K$ satisfying

$$A_t(\tilde{x}) = \tilde{x} \text{ for all integers } t \ge 1. \tag{1.3}$$

Denote by $\bar{\mathcal{M}}_*$ the closure of the set $\mathcal{M}_*$ in the uniform space $\mathcal{M}$. We consider the topological subspace $\bar{\mathcal{M}}_* \subset \mathcal{M}$ equipped with the relative topology and the metric d.

In this paper we study the asymptotic behavior of (unrestricted) infinite products of generic sequences of mappings belonging to the space $\bar{\mathcal{M}}_*$ and obtain convergence to a unique common fixed point. More precisely, we establish the following result, which generalizes the corresponding result in Reich and Zaslavski [3] (see also [4] and [5]). That result was obtained in the case where the set K was bounded.

Theorem 1.1. *There exists a set $\mathcal{F} \subset \bar{\mathcal{M}}_*$ which is a countable intersection of open and everywhere dense subsets of the complete metric space $(\bar{\mathcal{M}}_*, d)$ such that for each $\{B_t\}_{t=1}^{\infty} \in \mathcal{F}$, the following properties hold:*

(a) there exists a unique point $\bar{x} \in K$ such that $B_t(\bar{x}) = \bar{x}$ for all integers $t \ge 1$;

(b) if $t \ge 1$ is an integer and $y \in K$ satisfies $B_t(y) = y$, then $y = \bar{x}$;

(c) for each $\epsilon > 0$ and each $M > 0$, there exist a number $\delta > 0$ and a neighborhood $\mathcal{U}$ of $\{B_t\}_{t=1}^{\infty}$ in the metric space $\bar{\mathcal{M}}_*$ such that if $\{C_t\}_{t=1}^{\infty} \in \mathcal{U}$, $t \in \{1, 2, \dots\}$, and if $y \in B_K(\theta, M)$ satisfies $\rho(y, C_t(y)) \le \delta$, then $\rho(y, \bar{x}) \le \epsilon$;

(d) for each $\epsilon > 0$ and each $M > 0$, there exist a neighborhood $\mathcal{U}$ of $\{B_t\}_{t=1}^{\infty}$ in the metric space $\bar{\mathcal{M}}_*$, a number $\delta > 0$ and a natural number q such that if $\{C_t\}_{t=1}^{\infty} \in \mathcal{U}$, $m \ge q$ is an integer, $r : \{1, \dots, m\} \to \{1, 2, \dots\}$, and if $\{x_i\}_{i=0}^{m} \subset K$ satisfies

$$\rho(x_0, \theta) \le M$$

and

$$\rho(C_{r(i)}(x_{i-1}), x_i) \le \delta, \ i = 1, \dots, m,$$

then

$$\rho(x_i, \bar{x}) \le \epsilon, \ i = q, \dots, m.$$

2. Proof of Theorem 1.1

Elements of the space $\mathcal{M}$ will occasionally be denoted by a boldface letters: $\mathbf{A} = \{A_t\}_{t=1}^{\infty}$, $\mathbf{B} = \{B_t\}_{t=1}^{\infty}$, $\mathbf{C} = \{C_t\}_{t=1}^{\infty}$, respectively.

Let $\mathbf{A} = \{A_t\}_{t=1}^{\infty} \in \mathcal{M}_*$ and $\gamma \in (0, 1)$. There exists a point $x_{\mathbf{A}} \in K$ such that

$$A_t(x_{\mathbf{A}}) = x_{\mathbf{A}} \text{ for all integers } t \ge 1. \tag{2.1}$$

For each integer $t \ge 1$ and each $x \in K$, set

$$A_{\gamma,t}(x) = (1 - \gamma)A_t(x) \oplus \gamma x_{\mathbf{A}}. \tag{2.2}$$

By (1.1), (2.1), and (2.2), for all integers $t \ge 1$ and all points $x, y \in K$,

$$\begin{aligned}&\rho(A_{\gamma,t}(x), A_{\gamma,t}(y))\\ &= \rho((1 - \gamma)A_t(x) \oplus \gamma x_{\mathbf{A}}, (1 - \gamma)A_t(y) \oplus \gamma x_{\mathbf{A}})\\ &\le (1 - \gamma)\rho(A_t(x), A_t(y)) \le (1 - \gamma)\rho(x, y)\end{aligned} \tag{2.3}$$

and

$$A_{\gamma,t}(x_{\mathbf{A}}) = x_{\mathbf{A}}. \tag{2.4}$$

In view of (2.2–2.4),

$$\mathbf{A}_{\gamma} := \{A_{\gamma,t}\}_{t=1}^{\infty} \in \mathcal{M}_*. \tag{2.5}$$

Let n be a natural number. Fix a number

$$r(\mathbf{A}, n) > n + 2 + \rho(\theta, x_{\mathbf{A}}), \tag{2.6}$$

a number

$$M(\mathbf{A}, n) > r(\mathbf{A}, n) + \rho(\theta, x_{\mathbf{A}}) + 2, \tag{2.7}$$

a positive number

$$\delta(\mathbf{A}, \gamma, n) < (8n)^{-1}\gamma \tag{2.8}$$

and an integer

$$q(\mathbf{A}, \gamma, n) > 4 + 4nr(\mathbf{A}, n)\gamma^{-1}. \tag{2.9}$$

There exists an open neighborhood $V(\mathbf{A}, \gamma, n)$ of $\{A_{\gamma,t}\}_{t=1}^{\infty}$ in $\bar{\mathcal{M}}_*$ such that

$$\begin{aligned}V(\mathbf{A}, \gamma, n) \subset \{\{B_t\}_{t=1}^{\infty} \in \mathcal{M}:\\ (\{B_t\}_{t=1}^{\infty}, \{A_{\gamma,t}\}_{t=1}^{\infty}) \in \mathcal{U}(M(\mathbf{A}, n), \delta(\mathbf{A}, \gamma, n))\}.\end{aligned} \tag{2.10}$$

Assume that

$$\{C_t\}_{t=1}^{\infty} \in V(\mathbf{A}, \gamma, n), \tag{2.11}$$

$$m \ge q(\mathbf{A}, \gamma, n) \tag{2.12}$$

is an integer,

$$r : \{1, \ldots, m\} \to \{1, 2, \ldots\}, \tag{2.13}$$

and that a sequence $\{x_i\}_{i=0}^{m} \subset K$ satisfies

$$\rho(x_0, \theta) \le n \tag{2.14}$$

and

$$\rho(C_{r(i)}(x_{i-1}), x_i) \le \delta(\mathbf{A}, \gamma, n), \ \ i = 1, \ldots, m. \tag{2.15}$$

We now show by induction that for all integers $i = 0, \ldots, m$,

$$\rho(x_i, x_{\mathbf{A}}) \le r(\mathbf{A}, n), \tag{2.16}$$

$$\rho(x_i, \theta) \le M(\mathbf{A}, n) \tag{2.17}$$

and if $i < m$, then

$$\rho(x_{i+1}, x_{\mathbf{A}}) \le (1-\gamma)\rho(x_i, x_{\mathbf{A}}) + 2\delta(\mathbf{A}, \gamma, n). \tag{2.18}$$

Assume that $p \in \{0, \ldots, m-1\}$, (2.16) and (2.17) hold for all $i = 0, \ldots, p$ and that (2.18) holds for all nonnegative integers $i < p$. [Note that in view of (2.6), (2.7), and (2.14), our assumption holds for $p = 0$]. It follows from (2.3), (2.4), and (2.15) that

$$\begin{aligned}
\rho(x_{p+1}, x_{\mathbf{A}}) &\le \rho(x_{p+1}, C_{r(p+1)}(x_p)) + \rho(C_{r(p+1)}(x_p), x_{\mathbf{A}}) \\
&\le \delta(\mathbf{A}, \gamma, n) + \rho(C_{r(p+1)}(x_p), x_{\mathbf{A}}) \\
&\le \delta(\mathbf{A}, \gamma, n) + \rho(C_{r(p+1)}(x_p), \\
&\quad A_{\gamma, r(p+1)}(x_p)) + \rho(A_{\gamma, r(p+1)}(x_p), x_{\mathbf{A}}) \\
&\le \delta(\mathbf{A}, \gamma, n) + \rho(C_{r(p+1)}(x_p), A_{\gamma, r(p+1)}(x_p)) \\
&\quad + (1-\gamma)\rho(x_p, x_{\mathbf{A}}).
\end{aligned} \tag{2.19}$$

By (2.17), which holds for $i = p$, (1.2), (2.10), and (2.11),

$$\rho(C_{r(p+1)}(x_p), A_{\gamma, r(p+1)}(x_p)) \le \delta(\mathbf{A}, \gamma, n). \tag{2.20}$$

Relations (2.19) and (2.20) imply that

$$\rho(x_{p+1}, x_{\mathbf{A}}) \le (1-\gamma)\rho(x_p, x_{\mathbf{A}}) + 2\delta(\mathbf{A}, \gamma, n). \tag{2.21}$$

Thus, (2.18) holds for $i = p$. It follows from (2.16), which holds for $i = p$, (2.6), (2.8), and (2.21) that

$$\begin{aligned}
\rho(x_{p+1}, x_{\mathbf{A}}) &\le (1-\gamma)r(\mathbf{A}, n) + 2\delta(\mathbf{A}, \gamma, n) \\
&\le (1-\gamma)r(\mathbf{A}, n) + 2^{-1}\gamma \le r(\mathbf{A}, n).
\end{aligned}$$

By the above relation and (2.7),

$$\begin{aligned}
\rho(x_{p+1}, \theta) &\le \rho(x_{p+1}, x_{\mathbf{A}}) + \rho(x_{\mathbf{A}}, \theta) \\
&\le r(\mathbf{A}, n) + \rho(x_{\mathbf{A}}, \theta) \le M(\mathbf{A}, n).
\end{aligned}$$

Hence (2.16) and (2.17) hold for $i = p+1$ and the assumption made for p also holds for $p+1$. Therefore, our assumptions hold for $p = m$, (2.16) and (2.17) hold for all $i = 0, \ldots, m$, and (2.18) holds for all $i = 0, \ldots, m-1$.

We claim that for all $i = q(\mathbf{A}, \gamma, n), \ldots, m$,

$$\rho(x_i, x_{\mathbf{A}}) \le n^{-1}. \tag{2.22}$$

First we show that there exists $i \in \{0, \ldots, q(\mathbf{A}, \gamma, n)\}$ such that (2.22) holds.

Assume the contrary. Then

$$\rho(x_i, x_{\mathbf{A}}) > n^{-1}, \ \ i = 0, \ldots, q(\mathbf{A}, \gamma, n). \tag{2.23}$$

By (2.8), (2.18), and (2.23), for all integers $i = 0, \ldots, q(\mathbf{A}, \gamma, n) - 1$,

$$\begin{aligned}
&\rho(x_i, x_{\mathbf{A}}) - \rho(x_{i+1}, x_{\mathbf{A}}) \\
&\ge \gamma\rho(x_i, x_{\mathbf{A}}) - 2\delta(\mathbf{A}, \gamma, n) \\
&\ge \gamma n^{-1} - 2\delta(\mathbf{A}, \gamma, n) \ge \gamma(2n)^{-1}.
\end{aligned}$$

In view of the above inequality and (2.16),

$$\begin{aligned}
&r(\mathbf{A}, n) \ge \rho(x_0, x_{\mathbf{A}}) \ge \rho(x_0, x_{\mathbf{A}}) - \rho(x_{q(\mathbf{A},\gamma,n)}, x_{\mathbf{A}}) \\
&= \sum_{i=0}^{q(\mathbf{A},\gamma,n)-1} (\rho(x_i, x_{\mathbf{A}}) - \rho(x_{i+1}, x_{\mathbf{A}})) \ge q(\mathbf{A}, \gamma, n)\gamma(2n)^{-1}
\end{aligned}$$

and so,

$$q(\mathbf{A}, \gamma, n) \le 2nr(\mathbf{A}, n)\gamma^{-1}.$$

This contradicts (2.9). The contradiction we have reached proves that there indeed exists an integer $j \in \{0, \ldots, q(\mathbf{A}, \gamma, n)\}$ such that

$$\rho(x_j, x_{\mathbf{A}}) \le n^{-1}. \tag{2.24}$$

Next we claim that (2.2) holds for all integers $i \in \{j, \ldots, m\}$.

Indeed, by (2.24), inequality (2.22) is true for $i = j$. Now assume that $i \in \{j, \ldots, m\}$, $i < m$ and (2.22) holds. There are two cases:

$$\rho(x_i, x_{\mathbf{A}}) \le (2n)^{-1}; \tag{2.25}$$

$$\rho(x_i, x_{\mathbf{A}}) > (2n)^{-1}. \tag{2.26}$$

Assume now that (2.25) holds. In view of (2.8), (2.18), and (2.25),

$$\rho(x_{i+1}, x_{\mathbf{A}}) \le (1-\gamma)\rho(x_i, x_{\mathbf{A}}) + 2\delta(\mathbf{A}, \gamma, n)$$

$$\le (2n)^{-1} + 2\delta(\mathbf{A}, \gamma, n) \le n^{-1}.$$

Assume that (2.26) holds. Then it follows from (2.8), (2.18), (2.22), and (2.26) that

$$\begin{aligned}
\rho(x_{i+1}, x_{\mathbf{A}}) &\le (1-\gamma)\rho(x_i, x_{\mathbf{A}}) + 2\delta(\mathbf{A}, \gamma, n) \\
&= \rho(x_i, x_{\mathbf{A}}) - \gamma\rho(x_i, x_{\mathbf{A}}) \\
&\quad + 2\delta(\mathbf{A}, \gamma, n) \\
&\le n^{-1} - \gamma(2n)^{-1} + 2\delta(\mathbf{A}, \gamma, n) \le n^{-1}.
\end{aligned}$$

Thus, in both cases,

$$\rho(x_{i+1}, x_{\mathbf{A}}) \leq n^{-1}.$$

This means that we have shown by induction that (2.22) is indeed valid for all $i = q(\mathbf{A}, \gamma, n), \ldots, m$. Clearly, we have proved that the following property holds:

(P) For each

$$\{C_t\}_{t=1}^{\infty} \in V(\mathbf{A}, \gamma, n),$$

each integer $m \geq q(\mathbf{A}, \gamma, n)$, each

$$r : \{1, \ldots, m\} \to \{1, 2, \ldots\}$$

and each sequence $\{x_i\}_{i=0}^{m} \subset K$ which satisfies

$$\rho(x_0, \theta) \leq n$$

and

$$\rho(C_{r(i)}(x_{i-1}), x_i) \leq \delta(\mathbf{A}, \gamma, n), \ i = 1, \ldots, m,$$

we have

$$\rho(x_i, x_{\mathbf{A}}) \leq n^{-1}, \ i = q(\mathbf{A}, \gamma, n), \ldots, m.$$

Set

$$\mathcal{F} = \cap_{p=1}^{\infty} \cup \{V(\mathbf{A}, \gamma, n) : \mathbf{A} = \{A_t\}_{t=1}^{\infty} \in \mathcal{M}_*, \gamma \in (0, 1), \\ n \geq p \text{ is an integer}\}. \tag{2.27}$$

By (1.1), (2.1), and (2.2), for each $\mathbf{A} = \{A_t\}_{t=1}^{\infty} \in \mathcal{M}_*$, each $\gamma \in (0, 1)$, each integer $t \geq 1$ and each $x \in K$, we have

$$\begin{aligned}\rho(A_{\gamma,t}(x), A_t(x)) &= \rho((1-\gamma)A_t(x) \oplus \gamma x_{\mathbf{A}}, A_t(x)) \\ &\leq \gamma\rho(A_t(x), x_{\mathbf{A}}) \leq \gamma\rho(x, x_{\mathbf{A}}) \\ &\leq \gamma(\rho(x, \theta) + \rho(\theta, x_{\mathbf{A}})).\end{aligned} \tag{2.28}$$

In view of (1.2) and (2.28),

$$\{A_{\gamma,t}\}_{t=1}^{\infty} \to \{A_t\}_{t=1}^{\infty} \text{ as } \gamma \to 0^+ \text{ in } \bar{\mathcal{M}}_*.$$

When combined with (2.27), this implies that $\mathcal{F}$ is a countable intersection of open and everywhere dense subsets of $\bar{\mathcal{M}}_*$.

Assume that

$$\{B_t\}_{t=1}^{\infty} \in \mathcal{F} \tag{2.29}$$

and $M, \epsilon > 0$. Choose a natural number p such that

$$p > 8M + 8 \text{ and } (8p)^{-1} < \epsilon. \tag{2.30}$$

By (2.27) and (2.29), there exist

$$\mathbf{A} = \{A_t\}_{t=1}^{\infty} \in \mathcal{M}_*, \ \gamma \in (0, 1) \text{ and an integer } n \geq p \tag{2.31}$$

such that

$$\{B_t\}_{t=1}^{\infty} \in V(\mathbf{A}, \gamma, n). \tag{2.32}$$

Let

$$x \in B_K(\theta, M), \tag{2.33}$$

let $t \geq 1$ be an integer and consider the sequence $\{B_t^i(x)\}_{i=0}^{\infty}$. By (2.30)–(2.33) and property (P) (applied to $\{C_s\}_{s=1}^{\infty} = \{B_s\}_{s=1}^{\infty}$ and $r(j) = t, j = 1, 2, \ldots$), for all integers $i \geq q(\mathbf{A}, \gamma, n)$, we have

$$\rho(B_t^i(x), x_{\mathbf{A}}) \leq n^{-1} < \epsilon. \tag{2.34}$$

Since ϵ is an arbitrary positive number, we conclude that for each point $z \in B_K(\theta, M)$ and each integer $t \geq 1$, $\{B_t^i(z)\}_{i=0}^{\infty}$ is a Cauchy sequence. Since M is any positive number, we see that for each integer $t \geq 1$ and each $z \in K$, there exists

$$\lim_{i\to\infty} B_t^i(z)$$

in (X, ρ). In view of (3.34), for every integer $t \geq 1$ and every $z \in B_K(\theta, M)$,

$$\rho(\lim_{i\to\infty} B_t^i(z), x_{\mathbf{A}}) \leq \epsilon.$$

This implies that for each pair of points $z_1, z_2 \in B_K(\theta, M)$ and for each pair of natural numbers t_1, t_2,

$$\rho(\lim_{i\to\infty} B_{t_1}^i(z_1), \lim_{i\to\infty} B_{t_2}^i(z_2)) \leq 2\epsilon.$$

Since ϵ, M are arbitrary positive numbers, we may conclude that for each pair of integers $t_1, t_2 \geq 1$ and each pair of points $z_1, z_2 \in K$,

$$\lim_{i\to\infty} B_{t_1}^i(z_1) = \lim_{i\to\infty} B_{t_2}^i(z_2).$$

Let $\bar{x} \in K$ be such that

$$\bar{x} = \lim_{i\to\infty} B_t^i(z) \text{ for all } z \in K \text{ and all integers } t \geq 1. \tag{2.35}$$

In view of (2.35),

$$B_t(\bar{x}) = \bar{x} \text{ for all integers } t \geq 1. \tag{2.36}$$

It immediately follows from (2.35) and (2.36) that properties (a) and (b) hold. We claim that property (c) also holds.

Let

$$\{C_t\}_{t=1}^{\infty} \in V(\mathbf{A}, \gamma, n), \ t \in \{1, 2, \ldots, \}, \ y \in B_K(\theta, M)$$

and assume that

$$\rho(y, C_t(y)) \leq \delta(\mathbf{A}, \gamma, n). \tag{2.37}$$

Set

$$\begin{aligned} y_t &= y, \ t = 0, 1, \ldots, \\ r(i) &= t, \ i = 1, 2, \ldots. \end{aligned} \tag{2.38}$$

It follows from (2.37) and (2.38) that for all integers $t \geq 1$,

$$\rho(y_i, C_{r(i)}(y_{i-1})) = \rho(y, C_t(y)) \leq \delta(\mathbf{A}, \gamma, n). \tag{2.39}$$

By (2.30), (2.31), (2.37–2.39) and property (P) applied to any integer $m \geq q(\mathbf{A}, \gamma, n)$ and $x_i = y_i$, $i = 0, \ldots, m$,

$$\rho(y_i, x_{\mathbf{A}}) \leq n^{-1}, \; i = q(\mathbf{A}, \gamma, n), \ldots, m,$$

and

$$\rho(y, x_{\mathbf{A}}) \leq n^{-1}. \tag{2.40}$$

In view of (2.30), (2.31), (2.34), (2.35), and (2.40),

$$\rho(y, \bar{x}) \leq \rho(y, x_{\mathbf{A}}) + \rho(x_{\mathbf{A}}, \bar{x}) \leq 2n^{-1} < \epsilon. \tag{2.41}$$

Thus, property (c) does hold, as claimed.

Finally, we show that property (d) holds too. It follows from (2.34) and (2.35) that

$$\rho(x_{\mathbf{A}}, \bar{x}) \leq n^{-1}.$$

Assume that

$$\{C_t\}_{t=1}^{\infty} \in V(\mathbf{A}, \gamma, n),$$

let $m \geq q(\mathbf{A}, \gamma, n)$ be an integer, $r : \{1, \ldots, m\} \to \{1, 2, \ldots\}$, and let $\{x_i\}_{i=0}^{m} \subset K$ satisfy

$$\rho(x_0, \theta) \leq M$$

and

$$\rho(C_{r(i)}(x_{i-1}), x_i) \leq \delta(\mathbf{A}, \gamma, n), \; i = 1, \ldots, m.$$

By the relations above and property (P),

$$\rho(x_i, x_{\mathbf{A}}) \leq n^{-1}, \; i = q(\mathbf{A}, \gamma, n), \ldots, m. \tag{2.42}$$

It now follows from (2.30), (2.31), (2.41), and (2.42) that for all integers $i = q(\mathbf{A}, \gamma, n), \ldots, m$,

$$\rho(x_i, \bar{x}) \leq \rho(x_i, x_{\mathbf{A}}) + \rho(x_{\mathbf{A}}, \bar{x}) \leq 2n^{-1} < \epsilon.$$

Thus, property (d) indeed holds. This completes the proof of Theorem 1.1.

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